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Significant Precision in Crystal Structural Details: Holly Springs Hydroxyapatite

BY K. SUDARSANAN AND R. A. YOUNG

Georgia Institute of Technology, Atlanta, Georgia 30332, *U.S.A.*

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The degree to which physically significant precision in crystal-structural details can be obtained with selected routine procedures has been assessed and some illustrative applications to analyses of bonding effects and of impurity substitutions have been made. Least-squares refinements with X-ray data for three single crystals of mineral hydroxyapatite and with neutron diffraction data for a fourth of the same origin yielded $R \simeq 2\%$ in each case (≥ 40 parameters adjusted and > 500 reflections). Seemingly minor extinction corrections improved the mutual agreement among separate measures of some parameters from 4σ to the final 1σ (typically $< 5\%$) found for all, even anisotropic thermal, parameters in the X-ray cases. Final results were insensitive to reasonable changes in the weighting scheme. Comparisons of X-ray and neutron results, which generally agreed within 2σ, showed systematic differences associated with the oxygen atoms bonded to phosphorus in the phosphate group. Analysis of the final R-value also suggested (1) real differences, among the crystals, smaller than the σ's associated with individual parameters and (2) either a systematic inadequacy of the refinement model, or similar residual systematic errors (such as thermal diffuse scattering contributions), in both neutron and X-ray data. Direct refinement for the degree of fluorine substitution for OH in $Ca_{10}(PO_4)_6(OH)_2$ led to the same result, 8 at.% substitution, with both X-ray and neutron data. Analysis of the decrements found with both X-ray and neutron data in the apparent site-occupancy factors for the Ca atoms showed that a simple substitution of Mg^{2+} for Ca^{2+} at the same site is not in itself a sufficient substitutional model for this case.

Introduction

The precision ordinarily obtained in crystal structure refinements, even 'precision structure refinements,' is far less than that which would seem to be intrinsically available with present single crystal diffractometers used in a well-chosen routine fashion. The ACA Single Crystal Intensity Data Project (Abrahams, Alexander, Furnas, Hamilton, Ladell, Okaya, Young & Zalkin, 1967) has shown how well several experimental groups, each interested in routine precision measurements of intensities, agreed on the measurements of the same reflections from the same crystal (generally 3 to 5%). An I.U.Cr. project of similar name undertook to determine how well a greater variety of research groups, operating in approximately their normal fashion, agreed on the measurements of intensities from crystals from the same batch. As recently reported (*Commission on Crystallographic Apparatus*, 1966), the agreement in this second project is presently much poorer, *e.g.* no better than approximately 7% within a sub-set of most-similar results. Since X-ray apparatus is, presumably, stable to greater than 1%, and since counting statistical errors can also be small, it is clear that there must be very significant differences in the details of the techniques (and, in the I.U.Cr. project case, perhaps the crystal sizes, shapes, and states of twinning) used by the various groups. In the context provided by these projects, it is of interest to investigate how well a particular group can reproduce its own results in a series of independent experiments using different specimens as nearly identical as possible in composition (and, hence, in structural details) but not necessarily in size or mosaic spread.

A more important reason for the assessment of precision is that a number of questions of far reaching importance in the physics and chemistry of solids appear to lend themselves to study by precision crystallography. These include (1) anharmonicity in thermal motions (Willis, 1963, 1965), (2) actual electron wave functions in crystals (McWeeny, 1951, Freeman, 1959), (3) character, degree of direction and electron content of bonds (Dawson, 1964; McWeeny, 1951; Brill, 1950; Coppens, 1968*a*,*b*), and (4) structural location and role of impurity atoms in real crystals. In brief, precision crystallography should be expected to provide detailed determinations of structural features of real crystals, as opposed to those of the idealized crystals.

It now appears that Zachariasen's (1963, 1967) extinction-correction method has removed one of the major barriers to a reasonably successful reduction of intensities to $|F|^2$ values for many crystals. Newly useful comparisons of $|F|^2$ values from different crystals can thus be made, permitting examination of the real precision in physically significant quantities such as the coordinate parameters, temperature factors, and atom multipliers obtained from least-squares refinements. Physical interpretations may then be based on differences lying outside the limits of this demonstrated precision. With these limits established, bonding and impurity studies, for example, may then proceed both directly and indirectly, the latter through determination of the differences among specimens in

respect to distortions of both actual and apparent thermal motions. Significant precision in anisotropic temperature factors is, therefore, not only a good test of refinement precision but also a practically useful tool. Finally, the values obtained by X-ray means for the various physical quantities studied may be compared with those obtained from neutron diffraction, in which many of the systematic errors would be different, for some indication of their possible accuracy. The question of significance in the apparent precision, and something of accuracy, has been attacked here through comparison of results from X-ray studies of three distinct specimens and neutron studies of a fourth specimen, all of the same origin. The question of permissible physical interpretations has then been partially explored in an effort to determine further the potential value of precision crystallography.

No studies strictly comparable to the present one have been found in the literature. A somewhat similarly conducted assessment of precision has been made by Abrahams for X-ray studies of NaCl (Abrahams, 1964; Abrahams & Bernstein, 1965). The only adjustable parameters were the two isotropic temperature factors and the agreement among the results for the five single crystals studied was, generally, within about 5σ, where σ was about 4% of the quantity being determined. Both X-ray and neutron studies of lithium tantalate have been made by Abrahams, and coworkers (Abrahams & Bernstein, 1967; Abrahams, Hamilton & Sequeira, 1967), who found the positional coordinates agreed to within about one σ ($\sim 0{\cdot}003$) while the thermal parameters disagreed by as much as 50% ($\sim 2\sigma$). Thorium pentahydrate has been independently analyzed by neutron diffraction (Taylor, Mueller & Hitterman, 1966) and by X-ray diffraction (Ueki, Zalkin & Templeton, 1966) with the result that coordinate parameters agreed within about 0·003 ($\sim 3\sigma$) while the agreement for the thermal parameters was poorer. Calcium tungstate has been studied both by X-ray diffraction (Zalkin & Templeton, 1964) and by neutron diffraction (Kay, Frazer & Almodovar, 1964) with the result that the three independent coordinate parameters agreed within σ ($\sim 0{\cdot}001$) while some of the temperature factors disagreed by more than 25% ($\sim 4\sigma$). Finally, both Trueblood (1967) and Coppens (1968*a*,*b*) have recently reported that the temperature factors obtained from X-ray data consistently exceed those from neutron data, especially for the light atoms in organic crystals.

Experimental technique

For the X-ray studies three spherical single-crystal specimens of 0·17, 0·21, and 0·18 mm radius, respectively, were separately prepared from the same Holly Springs source of mineral hydroxyapatite. For this previously studied material the space group is $P6_3/m$ (Kay, Young & Posner, 1964; Posner, Perloff & Diorio, 1958). The values $a = 9{\cdot}424$ and $c = 6{\cdot}879$ Å were found to be satisfactory for predetermination of X-ray reflection angles and, hence, are probably correct to within 0·004 Å for these specimens. Intensity data were collected with a punched-tape controlled single-crystal diffractometer and filtered Mo $K\alpha$ radiation. The scanning range required had been determined previously by a brief manual survey. A survey run was then made under automatic control at the rate of ~ 700 reflections per day. This survey provided information from which the final data-collection program was prepared with scanning speeds and background counting ranges adjusted to yield $\sim 1\%$ counting statistics in the net intensity, subject to a maximum scanning time of 14 minutes. Those reflections which showed obviously abnormal character on the strip-chart recording, used for 100% visual monitoring of the data, were discarded.

Single-filter 2θ scans were used in the range $25° \leq 2\theta \leq 100$. For one specimen (X–23–4), 60 reflections were measured with balanced-filter ω-scans in the range $2\theta < 25°$. In the 2θ-scans, backgrounds were measured only on the high-angle side for $2\theta < 60°$ and on both sides of the peak for larger 2θ. In the ω-scans, four measures were required to establish the background (Young, 1965). A particular 'standard' reflection was remeasured every two or three hours. If two such successive measures did not agree within 1% the intervening data were discarded. Experience over several years with the reproducibility of intensities of various reflections from many specimens of Holly Springs hydroxyapatite has indicated that no radiation damage effects, from the incident beam, should be expected in the data. Thus, only one reflection was ordinarily used as standard.

Absorption corrections were based on the tabular data in *International Tables for X-ray Crystallography* (1959). The polarization factor for the kinematic case only was used in the initial data reduction.

Multiple-reflection errors were directly assessed in two ways and are thought not to be serious. The intensities of each of 30 reflections for which, finally, $||F_o|^2 - |F_c|^2| \geq 3\sigma$ were monitored as the crystal was rotated $\sim 20°$ about the particular diffraction vector. Generally, the visible multiple-reflection effects were not strong, though in one case (112) a 5% decrease in intensity was noted at the approximate orientation used for data collection. A second test was made by re-collection, with differing specimen orientation about the diffraction vector, of 65 reflections of mixed strong and weak character. Using subscripts to indicate the data set, we found $(I_1 - I_2)/(\sigma_1^2 + \sigma_2^2)^{1/2} \geq 3$ for 7 cases but $< 5{\cdot}3$ for all. Finally, comparisons of $|F_o|^2$ and $|F_c|^2$ show no consistent excess of $|F_o|^2$ over $|F_c|^2$ for the weak reflections, as might have been expected if multiple reflection effects were an important source of error.

Progress of structure refinements

Table 1 shows the weighted and non-weighted R values at different stages of refinement. X–23–4, X–23–6, and

X–23–10 refer to the X-ray data collected on three different crystals. For all but the final cycles, least-squares refinements on $|F|^2$ and an additional cycle on $|F|$ were carried out with an Algol version (Gallaher & Kay, 1964) of the full-matrix Busing, Martin & Levy (1962) program. For the X-ray cases, the hydrogen parameters were kept fixed at the values given by neutron diffraction. Atomic scattering factors for Ca^{2+}, P^+ and O^- were taken from *International Tables for X-ray Crystallography* (1962). For these refinement cycles the Ca^{2+} values were modified by the real part of the anomalous dispersion term given in the same Tables. The various R values used are defined as

$$R_n = \Sigma \, ||F_o|^n - s^n|F_c|^n| / \Sigma \, |F_o|^n \, ,$$
$$wR_n = \{\Sigma \, [w^{1/2}(|F_o|^n - |F_c|^n)]^2 / \Sigma \, [w^{1/2}|F_o|^n]^2\}^{1/2} \, , \qquad (1)$$

where n is either 1 or 2 and w is the reciprocal variance, as discussed later. After several cycles of refinement the R_2 values were those shown in the 'Before extinction correction' columns of Table 1. At this stage the observed structure factors were corrected for secondary extinction with Zachariasen's approximate method (Zachariasen, 1963).

The mutual agreement factors, R_M, between sets of optimally-scaled extinction-corrected $|F|^2$ values for the different crystals were then found to be about 3% where

$$R_M = \sum_H ||F_{H1}|^2 - |F_{H2}|^2| / \sum_H |F_{H1}|^2 \qquad (2)$$

and where $|F_{H1}|^2$ and $|F_{H2}|^2$ refer to the same reflections, with indices indicated by H, as observed with X-rays from two different crystals. In view of this good agreement (in $|F|^2$, *not* I), those reflection data seriously compromised by erratic instrument performance (*e.g.* a scaler digit being dropped in the read-out) were culled out by requiring that the various mutually-scaled and extinction-corrected measures of the same $|F^2|$ value agree to within 5σ, or the data for that reflection be discarded [σ given by equation (3)]. After the culling, which removed 6 reflections and changed the R_2 value by 0·1%, the extinction corrections were applied with the results shown in the 'After extinction correction' column of Table 1.

Since mineral apatites invariably show foreign ion substitution, atomic multiplying factors were also refined. Only the multipliers for Ca, O(H) and H showed shifts, from stoichiometric values, greater than one standard deviation. In subsequent refinements the other multipliers were therefore kept fixed at their stoichiometric values.

Various chemical analyses (Smith, 1967; Kay, Young & Posner, 1964; Mitchell, Faust, Hendricks & Reynolds, 1943) of Holly Springs hydroxyapatite have indicated the presence of fluorine to the extent of, variously, 0·16 to 0·28 wt.%. Other impurities, such as Mn and Mg were reported to be present in much smaller amounts. Hence fluorine was introduced in the refinement model at $0,0,\frac{1}{4}$, *i.e.* in the same position as in fluorapatite. Strong correlation between the multiplying factor for this F and the O(H) thwarted their simultaneous refinement with the X-ray data. Since it was expected that F^- was substituted for OH^-, the O(H) multiplier and all other variable parameters were successively refined with the F multiplier fixed at each of several values. With the F multiplier corresponding to 0·32 wt.%, wR_2 was effectively minimized and the sum of the multipliers of O(H) and F equalled the stoichiometric value for O(H). Further, and perhaps more significantly, the O(H) multipliers agreed for the X-ray and the neutron cases and, in the neutron case, essentially the same value was independently obtained for the multipliers of both O(H) and H.

Finally, one last refinement cycle was carried out for each X-ray specimen with the Johnson (1966) version of the Busing, Martin & Levy program incorporating both real and imaginary parts of the anomalous dispersion terms for Ca and P. The effect of this incorporation of the additional anomalous dispersion terms was very small, changing less than one third of the parameters by as much as one unit in the fourth significant figure and none of them by as much as one standard deviation. Although the non-weighted R_2 value was reduced in each case (from 3·3 to 3·1% in the largest case, X–23–4) the weighted R_2 value was not changed by this final cycle.

Results of refinements

As Table 1 shows, the final weighted R_1 values (*i.e.* those based on $|F|$) for the three crystals were about 2%. Structure factor values are provided in the Appendix.

Table 1. *Reliability factors at various stages*

		wR_2 (%)				R_2 (%)				
Specimen	Number of reflections	Before extinction correction	After extinction correction	Final	R_1 (%) Final	Before extinction correction	After extinction correction	Final	R_1 (%) Final	Extinction parameter, C
X-23-4	726	4·9	4·6	3·9	2·0	3·8	3·6	3·1	2·3	0·0027
X-23-6	703	6·2	4·9	3·7	1·9	5·4	3·5	3·1	2·2	0·0030
X-23-10	500	5·1	4·8	4·2	2·2	3·6	3·5	3·3	2·8	0·0036
Neutron	253	11·8	4·8	4·5	2·5	16·5	4·4	4·1	2·3	0·0012

Table 2 gives the final determinations of the various atomic parameters for each of the four crystals. For the three X-ray cases, all of the separate determinations of the same parameter agree within one standard deviation, even for the cross-terms of the temperature factors. This excellent agreement of the parameters confirms that the estimated standard deviations, though small, are of correct magnitude.

The neutron diffraction data collected by Kay, Young, & Posner (1964) were here corrected anew for extinction by means of Zachariasen's expression with appropriate modifications, and the positional and thermal parameters and the atomic multipliers were refined. In comparing the neutron and X-ray results given in Table 2, one finds agreement within one standard deviation for all the positional parameters except that of the hydroxyl oxygen, O(H), for which the agreement is still within three standard deviations. Of the 26 independent thermal parameters, 20 agree within one standard deviation, three agree within two standard deviations and the remaining three agree within three standard deviations. However, even these small disagreements seem to show some systematic character that is physically reasonable, as will be discussed later.

Possible contributions to R

Although the final R values are 'good', one may legitimately ask why, with 1% statistics as the experimental goal for most reflections, the R_2 values were not still lower. Further, the question of the physical significance of differences in R values at this level arises. We therefore estimate the known contributions to R_1 and R_2; the balance must be the result of uncorrected short-term variations in overall performance of the diffraction instrument, inadequacy of the model and other unknown systematic errors.

(1) *Counting statistics*

The standard deviation, σ, in each net intensity due to errors in counting statistics was calculated with the relation

$$\sigma = C_1 + C_2|F|^2 \tag{3}$$

where C_1 is a constant representing the minimum detectable intensity and for the 2θ scan case (Young, 1965)

$$C_2 = \left[\frac{1+(1+t)/\mathscr{S}}{I_N}\right]^{1/2} \tag{4}$$

where I_N = net intensity measured, t = ratio of the time spent in measuring the peak intensity to that spent on background, and $\mathscr{S}$ = signal-to-noise ratio. For the balanced filter ω-scans the relation

$$C_2 = [I_1 + I_2 + (I_{b1} + I_{b2})t^2]^{1/2}/I_N \tag{5}$$

was used. I_1 and I_2 are the gross peak intensities while I_{b1} and I_{b2} are the background intensities (sampled at both sides of the peak) for ω-scans made with the first and second filter, respectively. The net intensity is given by $I_N = I_1 - I_2 - (I_{b1} - I_{b2})t$.

To find the possible contribution to R_2 from counting statistics we calculated the R_2 values to be expected as a result of an intensity measurement error of one standard deviation, *i.e.* $(\Sigma\sigma)/(\Sigma|F_o|^2)$ for the unweighted, and $(N^{1/2})/[\Sigma w(|F_o|^2)^2]^{1/2}$ for the weighted, case where N is the number of observations and w is the weight for each observation. Table 3 gives these calculated values, which indicate the most probable values of nonweighted and weighted R_2 factors that could be expected even if the model described the real structure perfectly and the data were free of all systematic errors.

(2) *Absorption correction*

The crystals used were ground to approximate spheres of radii 0·17, 0·21 and 0·18 mm with respective μR values of 0·49, 0·62, and 0·52. The maximum deviation of the crystal radii from the spherical value was approximately 0·01 mm. For these values of μR, uncertainty in the absorption correction due to a variation in the radii of approximately 0·01 mm is less than 0·6% in the worst case, and is therefore certainly less than 0·3% on the average.

Treating the components of R_2 in Table 3 as independent random variables indicates that in each case a substantial contribution to R_2 remains unaccounted for by the known random errors. Presumably the quantities in the 'Remaining component' columns in Table 3 are then due in large part to the combined effects of erratic machine operation (thought to be $<0{\cdot}5\%$, as assessed by separate reproducibility tests), the degree to which the model fails to represent the real crystal by reason of oversimplification, and residual systematic errors common to both the X-ray and neutron cases. An obvious source of such systematic error is the thermal diffuse scattering (TDS) contribution to apparent Bragg intensities (Young, 1965; Nilsson, 1957, 1959). Although it is clearly needed, no correction for TDS contribution has been undertaken in the present work.

Sensitivity of R and parameters to various factors

Weighting scheme

So much has been written about weighting schemes that one tends to feel the choice must be important, perhaps even in the final stages of refinement. The weight for each reflection was estimated from the calculated standard deviation and the function minimized was $\Sigma w(|F_o|^2 - |F_c|^2)^2$. In all cases $\Sigma w(\Delta|F|^2)^2/(m-n)$ was found to be more than 2·5, where $\Delta|F|^2 = |F_o|^2 - |F_c|^2$, m is the total number of observations and n the number of parameters varied. As a check on the weights used, the complete set of $|F_o|^2$ values was divided into 12 ranges in $|F_o|^2$ and the average values of $w(|F_o|^2 - |F_c|^2)$ in each range were plotted against the corresponding average values of $|F_o|^2$. The plot was

concave upward. The standard deviation for each observation was then revised to

$$\sigma' = C_1 + C_2|F|^2 + C_3 + C_4|F|^2 , \qquad (6)$$

where the constants C_3 and C_4 were assigned values such that the revised weights gave a 'fairly good' straight line of zero slope for the plot $w(|F_o|^2 - |F_c|^2)_{av}$ *versus* $|F_o|^2_{av}$. In subsequent least-squares refinements with the altered weights it was found that the shift in the parameters was less than one standard deviation. There was no change $>0{\cdot}1\%$ in the unweighted R_2 values but the weighted R_2 factors increased, in the worst case, from 4·6% to 5·0% and $\Sigma\, w(\Delta|F|^2)^2/(m-n)$ was finally found to be between 1·4 and 1·6. The change in weighting was, in general, different for each reflection. However, since the unweighted R_2 values were not changed significantly, some kind of 'average' fractional change in weights is indicated by the change in $\Sigma w(\Delta|F|^2)^2/(m-n)$, *i.e.* $\sim$40%. The results reported in Table 2 came from refinements based on the standard deviations calculated with equation (3).

Extinction corrections

The secondary extinction correction, though seemingly slight [*i.e.* changing $R_2(|F|^2)$ from 5·4% to 3·5% in the worst case] was important to the excellent agreement finally found; prior to application of the extinction correction some of the temperature parameters differed by more than four standard deviations. Table 4 shows explicitly the relatively large effect which small corrections had on some of the independent parameters, the hydroxyl oxygen being chosen for the example because it exhibited the largest effects.

Since the extinction correction differed among the crystals, yet iterative application of it brought about agreement among previously differing measures of the same parameters, the correction may be accounted successful, necessary, and correctly applied here.

Sensitivity of R to parameter differences

In view of the small R values, a natural question is to what extent are improvements in R associated with significant changes in parameters. The final parameters for cases X–23–6 and X–23–4 were interchanged and the R_2 value recalculated with the result that R_2 increased from 3·2% to 4·0% and 3·4% to 4·2% – rather large changes in the present context. By Hamilton's (1965) R-ratio test these changes would be significant at more than the 99·5% level. Thus it seems probable that some real differences do exist, probably both in the crystals themselves and in the systematic errors associated with each, and these are indicated in Table 2, even though their effects on the parameters are smaller than the statistical standard deviations in the individual refinements.

Physical interpretations of results

It is evident from Table 2 that the atoms are consistently located and the thermal parameters are consistently determined with precision by this set of structure refinements. That accuracy as well as precision has been approached with these X-ray analyses is suggested by the agreement between X-ray and neutron results; however, corrections for TDS and any other sources of systematic error common to both techniques would have to be made before accuracy could be claimed. Nonetheless, the precision obtained and the fraction of the final R *not* due to statistical factors would seem to be possibly sufficient to support further examination of the thermal motions for anharmonic character such as has concerned Willis (1963, 1965) in UO_2 and CaF_2. Possible asymmetry of form factors might also be investigated by way of an apparent anharmonic contribution to the thermal parameters (Dawson, 1964). However, no such examinations have been undertaken here, one reason being that for such purposes the data could usefully be made still better by the use of longer counting times.

An apparent effect of bonding character does occur in Table 2. First, one notices that the thermal parameters of the oxygen atoms in the phosphate group are consistently measured larger with X-rays than with neutrons, whereas for the other atoms there does not appear to be a consistent pattern of difference. Since the P–O bonds are the only ones present which would be expected to have substantial covalent character, it is in the thermal parameters of these oxygen atoms, in particular, that one might then expect most easily to see neutron *vs.* X-ray differences due to the redistribution and probable anisotropy of the electron density associated with bonding. Conversion of the temperature factors of the oxygen atoms to thermal ellipsoids was done with an Algol version (Gallaher & Taylor, 1964) of the Busing, Martin & Levy (1964) *Function and Error Program*. The principal-axis data are given in Table 5. For all three atoms the smallest principal axis lies along the P–O bond direction, as expected. However, both X-ray and neutron refinements yield essentially the same lengths for this principal axis. The X-ray *vs.* neutron differences shown in the β_{ij} values of O_{III} in Table 2 therefore actually arise from differences in the real or apparent thermal vibrations perpendicular to the P–O bond. For these directions the individual X-ray *vs.* neutron principal-axis results differ by $1{\cdot}4\sigma$ in one case, $\sim 2{\cdot}5\sigma$ in two cases and $\sim 4\sigma$ in three cases. Statistical significance of these differences is strongly enhanced by the fact that they are all in the same direction with an average difference of about 3σ.

This excess of apparent thermal motion perpendicular to the P–O bond in the X-ray case may occur because some experimental error has enlarged the apparent X-ray temperature factors or decreased the apparent neutron temperature factors, but it is not obvious why the oxygen atoms should be preferentially so affected. It seems more probable that this apparent excess may be physically interpretable in terms of

crystal-field effects or bonding effects, as described above, on the atomic wave functions. However, a thorough investigation of that point is left for future work in which the comparisons can be based on sets of data for which internal consistency of several sets has also been shown for the neutron data, as it has been here for the X-ray data. It does appear that, compared to neutron results, this excess in the X-ray determined thermal parameter of the lighter atoms may become commonly observed, as both Trueblood (1967) and Coppens (1968*a*,*b*) have noted similar differences occurring for some organic crystals.

A second aspect which invites physical interpretation is the matter of deficiencies in certain ions as indicated by the final atom-multiplying factors, shown in Table 6. It is encouraging that the X-ray and neutron results agree so well. Yet, since the scattering powers of atoms differ for X-rays and neutrons in an irregular way, complete agreement among the multipliers would be physically realistic only in cases in which the correct ions in the correct proportions have been included in the refinement model. Such a situation would occur naturally only for the stoichiometric case. It can be produced in the non-stoichiometric case by explicit introduction of impurity ions, at their proper locations, into the refinement model, as was done here for F substitution for O(H). But even without explicit substitution in the model, the expected X-ray *vs.* neutron differences make possible some consistency tests of substitutional models postulated to account for specific deficiencies such as, in this case, the apparent Ca deficiency. A test may be devised as follows. In the refinement of the neutron data, consider the effect on the multiplier of atom 1 if a fraction, x, of its sites are filled instead with atom 2, the scattering lengths being b_1 and b_2, respectively. It is necessary that

Table 2. *Positional coordinates and thermal parameters of Holly Springs hydroxyapatite*

Values $\times 10^4$, standard deviations given in parentheses for parameters varied.

	x	y	z	β_{11}	β_{22}	β_{33}	β_{12}	β_{13}	β_{23}
O_I									
X-23-4	3284 (2)	4848 (2)	2500	39 (2)	30 (1)	54 (2)	27 (1)	0	0
X-23-6	3282 (2)	4846 (2)	2500	37 (2)	29 (2)	55 (2)	25 (1)	0	0
X-23-10	3282 (2)	4847 (2)	2500	40 (2)	29 (1)	54 (2)	26 (1)	0	0
Neutron	3282 (2)	4846 (1)	2500	35 (1)	28 (1)	44 (2)	24 (1)	0	0
O_{II}									
X-23-4	5873 (2)	4651 (2)	2500	20 (1)	26 (1)	96 (3)	9 (1)	0	0
X-23-6	5871 (2)	4649 (2)	2500	18 (1)	28 (2)	97 (2)	10 (1)	0	0
X-23-10	5872 (2)	4652 (2)	2500	19 (1)	28 (2)	102 (3)	11 (1)	0	0
Neutron	5876 (1)	4652 (1)	2500	20 (1)	24 (1)	89 (2)	9 (1)	0	0
O_{III}									
X-23-4	3437 (2)	2579 (1)	702 (2)	92 (2)	42 (1)	49 (2)	45 (2)	−41 (2)	−27 (1)
X-23-6	3434 (2)	2579 (2)	704 (2)	89 (2)	43 (2)	53 (2)	44 (2)	−38 (2)	−26 (1)
X-23-10	3438 (2)	2581 (2)	704 (2)	87 (2)	45 (2)	53 (2)	44 (2)	−42 (2)	−24 (2)
Neutron	3433 (1)	2579 (1)	704 (1)	84 (1)	39 (1)	44 (1)	43 (2)	−34 (2)	−21 (1)
P									
X-23-4	3987 (2)	3685 (1)	2500	19 (1)	17 (1)	25 (1)	10 (1)	0	0
X-23-6	3985 (2)	3684 (1)	2500	18 (1)	17 (1)	27 (1)	10 (1)	0	0
X-23-10	3987 (2)	3685 (1)	2500	20 (1)	17 (1)	27 (1)	11 (1)	0	0
Neutron	3983 (1)	3683 (1)	2500	18 (1)	20 (1)	26 (2)	9 (1)	0	0
Ca_I									
X-23-4	3333	6667	15 (1)	31 (1)	β_{11}	18 (2)	$\beta_{11}/2$	0	0
X-23-6	3333	6667	14 (1)	31 (1)	β_{11}	21 (2)	$\beta_{11}/2$	0	0
X-23-10	3333	6667	14 (1)	32 (1)	β_{11}	22 (2)	$\beta_{11}/2$	0	0
Neutron	3333	6667	13 (1)	33 (1)	β_{11}	24 (2)	$\beta_{11}/2$	0	0
Ca_{II}									
X-23-4	2468 (2)	9934 (1)	2500	21 (1)	23 (1)	28 (1)	10 (1)	0	0
X-23-6	2465 (1)	9933 (1)	2500	21 (1)	21 (1)	31 (1)	10 (1)	0	0
X-23-10	2468 (2)	9934 (1)	2500	21 (1)	24 (1)	30 (1)	10 (1)	0	0
Neutron	2465 (1)	9931 (1)	2500	24 (1)	25 (2)	30 (2)	12 (1)	0	0
O(H)									
X-23-4	0	0	1950 (7)	25 (2)	β_{11}	102 (8)	$\beta_{11}/2$	0	0
X-23-6	0	0	1960 (6)	26 (3)	β_{11}	95 (7)	$\beta_{11}/2$	0	0
X-23-10	0	0	1955 (8)	26 (3)	β_{11}	98 (8)	$\beta_{11}/2$	0	0
Neutron	0	0	1978 (7)	25 (2)	β_{11}	101 (8)	$\beta_{11}/2$	0	0
H									
X-23-4	0	0	608	129	β_{11}	104	$\beta_{11}/2$	0	0
X-23-6	0	0	608	129	β_{11}	104	$\beta_{11}/2$	0	0
X-23-10	0	0	608	129	β_{11}	104	$\beta_{11}/2$	0	0
Neutron	0	0	608 (14)	129 (8)	β_{11}	104 (12)	$\beta_{11}/2$	0	0

$$(1-x)b_1+xb_2=(1-y)b_1\,, \qquad (7)$$

where y is the fractional decrement in the multiplier of atom 1. (Note that y could be negative.) Then

$$x=y/[1-(b_2/b_1)]\,. \qquad (8)$$

A somewhat similar relation must hold for the X-ray results, but here b_2 and b_1 must be replaced by $\langle f_2\rangle$ and $\langle f_1\rangle$, the effective values of the atomic scattering factors f_2 and f_1 properly weighted and averaged over the sin θ/λ range used in the experiment.

According to the chemical analyses, some Mg is present in Holly Springs hydroxyapatite. As one example of how data of the type obtained from these refinements may be used to assess substitutional models, we test for the substitution of Mg for Ca at the same site. The scattering lengths are $b_{Ca}=0{\cdot}49\times 10^{-12}$ cm and $b_{Mg}=0{\cdot}35\times 10^{-12}$ cm. The arithmetic-mean value of the multiplying factor measured with X-rays is 0·3246 for Ca_I and 0·4828 for Ca_{II}. Over the range $0{\cdot}2\leq\sin\theta/\lambda\leq 0{\cdot}9$ the ratio $f_{Mg^{2+}}/f_{Ca^{2+}}$ runs from 0·65 to 0·40. For the sake of our, thus necessarily rough, calculation, we will use 0·5 for the effective average value of this ratio for our data. Table 7 shows the results for the degree of substitution as calculated from equation (8).

Table 3. *Apparent components of R_2 and wR_2 values*

	From counting statistics			Observed final values		Remaining component	
	$\frac{\Sigma\sigma}{\Sigma\lvert F\rvert^2}$	$\frac{\sqrt{N}}{\sqrt{\Sigma w(\lvert F\rvert^2)^2}}$	From absorption	R_2	wR_2	R_2'	wR_2'
Specimen	(%)	(%)	(%)	(%)	(%)	(%)	(%)
X-23-4	2·0	2·1	0·3	3·1	3·9	2·3	3·3
X-23-6	1·2	1·2	0·3	3·1	3·7	2·8	3·5
X-23-10	1·9	2·0	0·3	3·3	4·2	2·7	3·7

Table 4. *Effect of small corrections as shown in independent parameters of* O(H)

	Parameters (σ) $\times 10^4$						
Refinement*	x	y	z	β_{11}	β_{33}	R_2 (%)	wR_2 (%)
X-23-6*A*	0	0	2037 (39)	19 (4)	462 (16)	5·4	6·2
X-23-6*B*	0	0	2028 (8)	29 (3)	100 (1)	3·6	4·6
X-23-6*C*	0	0	1969 (6)	26 (3)	94 (7)	3·2	3·7

* *A* No extinction correction.
B Extinction correction applied.
C Extinction correction applied, F impurity (0·32 wt %) introduced at fixed position, multipliers of Ca and O(H) varied.

Table 5. *Thermal ellipsoid axes*

r.m.s. values and (σ) in Å $\times 10^3$.

	Axis parallel to P–O		Axis along **z**		Axis ~ perpendicular to P–O and Ca–O		Remaining axis	
	X-ray	Neutron	X-ray	Neutron	X-ray	Neutron	X-ray	Neutron
O_I	68 (3)	67 (3)	114 (2)	103 (2)			120 (2)	113 (2)
O_{II}	82 (3)	82 (3)	153 (2)	146 (2)			101 (2)	96 (3)
O_{III}	73 (3)	76 (3)			192 (2)	180 (2)	101 (2)	90 (2)

Table 6. *Structural models for substitutions*

Multiplying factors (σ) $\times 10^4$

	Ca_I		Ca_{II}		O(H)	
Specimen	Before refinement	After refinement	Before refinement	After refinement	Before refinement	After refinement
X-23-4	3333	3234 (11)	5000	4852 (14)	1666	1525 (20)
X-23-6		3216 (13)		4870 (18)		1494 (24)
X-23-10		3259 (18)		4837 (24)		1515 (28)
Neutron		3226 (13)		4817 (14)		1541 (13)

Table 7. *Consistency check of substitutional model*

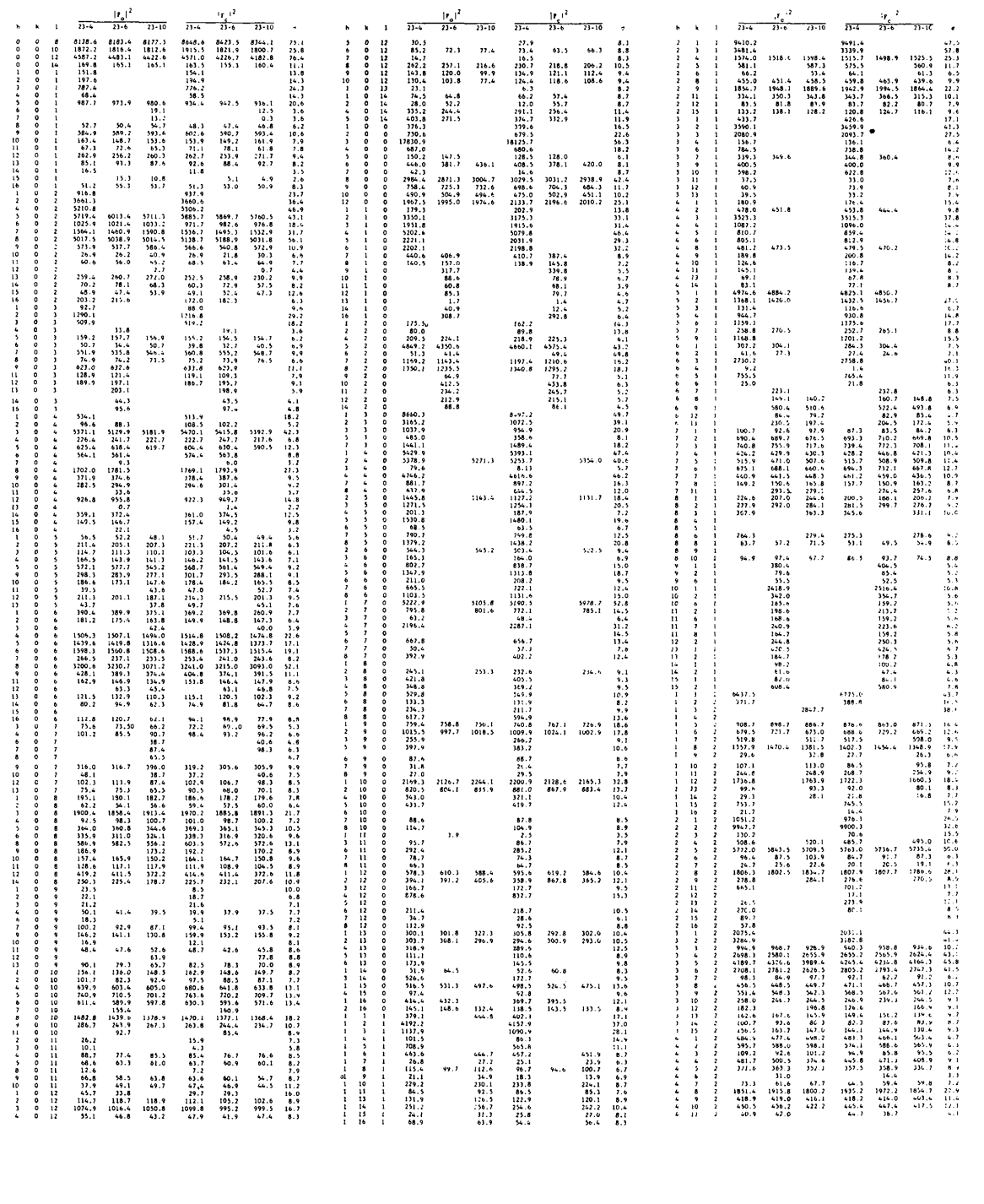
	Neutron case			X-ray case		
	y(%)	b_2/b_1	x(%)	y(%)	$\langle f_2\rangle/\langle f_1\rangle$	x(%)
Ca_I	3·21	0·715	11±2	3·01	0·5	6·0
Ca_{II}	3·66		12±2	2·96		5·9

However, even though very generous estimates of possible error in Table 7 are made, the X-ray-based and neutron-based estimates of x disagree. This failure of the consistency check is itself informative, for it may therefore be concluded that the observed Ca-

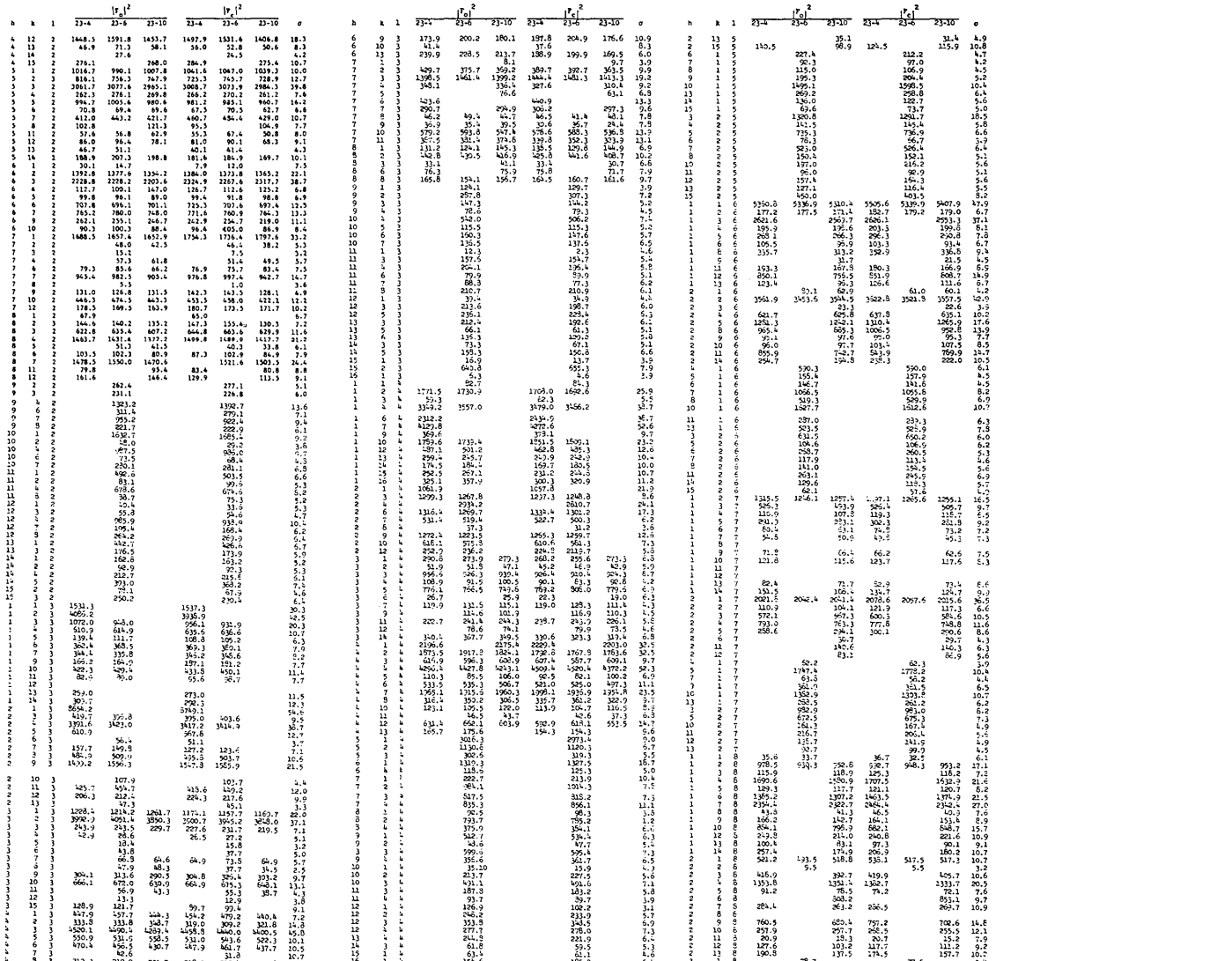
APPENDIX

Squares of structure amplitudes (X-ray)

Specimens to which the values refer are identified by numbers at the head of each column.



APPENDIX (*cont.*)



multiplier decrements do not arise from a simple substitution of Mg for Ca. While there may be some substitution of Mg for Ca at the same site, it is clear that other substitutions, perhaps at other sites, and probably of other atoms or even vacancies, must also exist in these specimens.

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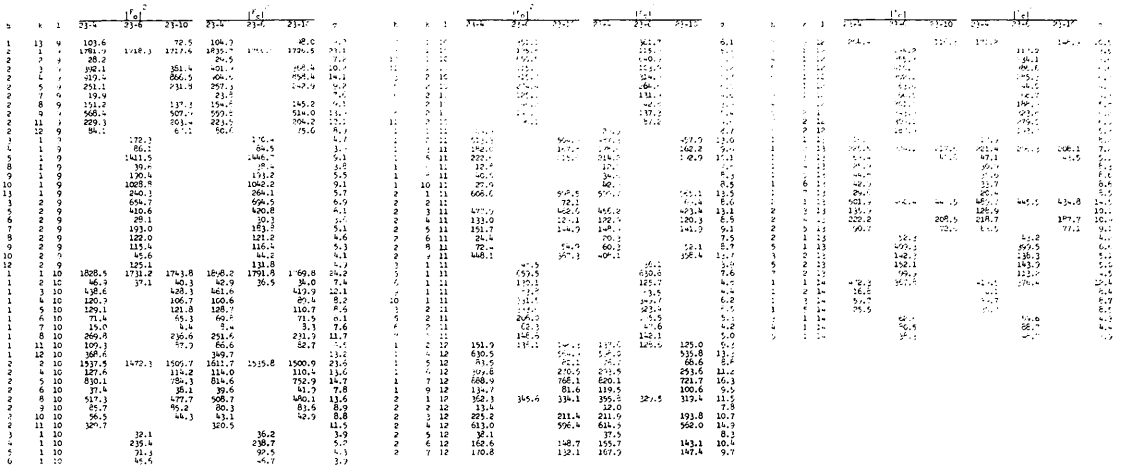
APPENDIX (*cont.*)



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